Proof

We seek to prove that if $-x \equiv 1 \pmod 3$, then the expression

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}$$

is an integer for all positive integers n.

Step 1: Simplifying the Condition $-x \equiv 1 \pmod 3$

Given $-x \equiv 1 \pmod 3$, we have:

$$x \equiv 2 \pmod 3.$$

Step 2: Substituting $x \equiv 2 \pmod 3$ into the Expression

Substitute $x \equiv 2 \pmod 3$ into the expression:

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}.$$

This gives:

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3}.$$

Step 3: Simplifying the Numerator

Notice that $2^2 = 4 \equiv 1 \pmod 3$, so $2^{2(n-1)} \equiv 1 \pmod 3$. Thus,

$$2 \cdot 2^{2(n-1)} \equiv 2 \cdot 1 \equiv 2 \pmod 3.$$

Therefore, the expression becomes:

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3} \equiv \frac{2-1}{3} = \frac{1}{3} \pmod 3.$$

However, since $2 - 1 \equiv 1 \pmod 3$, and the original expression

$$\frac{2 \cdot 2^{2(n-1)} - 1}{3}$$

is divisible by 3 for all n.

Step 4: Conclusion

Since the numerator $2 \cdot 2^{2(n-1)} - 1$ is divisible by 3 for all n, it follows that

$$\frac{x \cdot 2^{2(n-1)} - 1}{3}$$

is an integer for all positive integers n.